THE FUNCTION CONCEPT IN HIGH SCHOOL MATHEMATICS[1]

By DR. J. M. KINNEY
Crane Junior College, Chicago

The Fundamental Character of Functionality. The perimeter of a square depends on the length of a side. The area of a circle depends on its radius. The time of vibration of a simple pendulum is related in a very definite way to the length of the pendulum. The distance passed over by a body moving with uniform velocity depends on the velocity and the time. The amount of a sum of money placed at compound interest depends on the principal, rate, and time. The size of the crops depend on the acreage, on heat, moisture, fertility of the soil, and the industry of the farmer. The rent one pays for his house depends on the size, improvements, location, inflation of the currency, and the conscience of the owner. It would be possible to enumerate thousands of instances in which things are related to things. The notion of relationship or dependence is one of the most, if not the most, fundamental notion of mathematics. It is usually designated as the principle of functionality. The notion of function has long been consciously recognized by mathematicians. Leibnitz used the word function to designate different kinds of geometrical magnitudes associated with a variable point on a curve. The meaning of the word has been broadened from time to time so that at present it implies relationships in which one object corresponds in some definite way to another object. Thus if we should choose x to be a variable representing any one of the hours today and y another variable representing the temperature, then we could set up a correspondence between the values of x and y in such a way that for any value of x there is a unique value of y. Under such restrictions y is said to be a function of x.

The Function Concept as an Organizing Principle. If the function concept is the most fundamental thing in mathematics, why should it not be used as an organizing principle? This question seems to have occurred first to Professor Felix Klein

[1] Read at the Chicago Meeting of the National Council of Mathematics Teachers, March 1, 1922.

and was expressed by him in a paper read before the International Congress of Mathematicians which met in Chicago in 1893.

Professor E. H. Moore, in an article on "The Cross-section Paper as a Mathematical Instrument" appearing in The School Review, May 1906, says: "In this note I wish to suggest possibilities, which may have escaped attention, in the systematic use of cross-section paper as a unifying element in mathematics. I know of no medium serving to bring together so easily the three phases, or dialects, of pure mathematics—*number, form, formula*—and to lead so directly to the concept of functionality —a concept which, since the seventeenth century, has dominated advanced mathematics and the sciences, a concept which in the twentieth century, according to the auspices, will play a fundamental roll in the reorganization of elementary mathematical education. Students will gain a more easy and perfect mastery of mathematics, and their work will be full of richer direct and indirect value for them, when the primary emphasis is laid on the recognition, the depiction, and the closer study of functional relations between variable quantities."

Since the appearance of these papers a number of books on first year college mathematics organized about the principle of functionality have been written in this country. It is only recently that such a book has appeared in the secondary field. When one beholds in the secondary texts on algebra the heterogeneous arrangement of topics—equations, factoring, exponents, progressions, logarithms, radicals, and imaginaries—one wonders why the attempt was not made long ago to organize secondary mathematics about this principle. Thanks to the earnest recommendation of the National Committee on Mathematical Requirements we shall no doubt soon see numerous attempts to so organize our courses.

In the matter of the construction of a high school course in mathematics let us first give our attention to an outline of the functional relations which might be included in such a course. The first relation to be considered must be of a very simple type. If the work of the seventh and eighth grades has not been organized along the lines which I am proposing I should begin the ninth year with material falling under the type $y = ax$.

Under this heading one can find an abundance of material including such topics as perimeters, measurement and approximate numbers, percentage and interest, scale drawings, similar triangles, trigonometric ratios, uniform motion, motion under constant acceleration, arithmetic progressions, linear variations, and graphs.

After this relation follows quite naturally the general linear type $y = b + ax$. Some of the topics considered under the simpler type may again be taken up. Thus the formula for uniform motion $d = rt$ may be extended to the form $d = d_o + rt$ or $d = d_o + (r_2 - r_1)\, t$. The formula for motion on an inclined plane, a special form of motion under constant acceleration, $v = 32\ t \sin i$, where i is the angle of inclination, may be extended to $v = v_o + 32\ t \sin i$. The formula $h = d \tan \alpha$ which gives the height of an object may be replaced by the formula $h + h_o = d \tan \alpha$. A large number of relations arising from business, social, and physical situations fall under this general linear type.

Under this type one could and probably would give attention to the fundamental operations on directed numbers.

Before passing the linear type we should give some attention to linear pairs. One can find a large amount of material from geometric, social, mechanical, and miscellaneous situations in which two linear relations are concerned.

We come now to the quadratic types. This chapter may be introduced with a consideration of the hyperbolic relation $z = axy$. Under this heading one could consider such topics as areas of parallelograms, triangles, trapezoids, including such formulas as $A = bh \sin \alpha$ and $A = \frac{1}{2}bh \sin \alpha$, work power, the sum of arithmetic series, and the inverse variation. The more general type $z = (x + a)\ (y + b)$ may be considered in connection with the question of the keeping of significant figures in the product of two approximate numbers.

The parabolic relation $y = ax^2$ is rich in applications to geometry and physics. Such topics as the areas of squares and circles, the area under the segment of the line $y = ax$ from o to x, the simple pendulum, work performed by a uniformly variable force such as that performed in stretching a spiral spring,

distance passed over by a body moving with constant acceleration and having an initial o-velocity, projectiles, and the variation as the square. The areas of such figures as the hollow square and the annulus, the area under the line segment $y = ax$ from x, to x_2, and the work performed by a uniformly varying force between the values f_1 and f_2 give rise to formulas of the type $y = a\ (x^2 - b^2)$. The type $y = a\ (x + b)^2$ arises from the consideration of changes in x in the relation $y = ax^2$.

The general parabolic type $y = ax^2 + bx + c$ arises in connection with problems on areas, maxima and minima, distances traveled by bodies moving under constant acceleration, and paths of projectiles. The formula for the sum of an Arithmetic Series falls under this type and may be used in solving many interesting problems. This type concludes the work of the ninth grade. The course which I have just outlined is one which I have been developing during the past four years. The problem material which is assumed to lie within the easy comprehension of the pupil arranges itself automatically under the various types of relations.

I have outlined in some detail the work of the ninth grade in order to give you a somewhat definite notion as to the manner in which the notion of functionality may be used by the teacher in ordering the course and in selecting problem material. I shall now speak briefly of the functions which may be considered during the remaining three years.

At the beginning of the second year a little time should be given again to the linear and quadratic functions. Determination of the constants in a linear relation from given data gives a review in the solution of linear equations. The notion of slope of a line and of a curve should be developed. In connection with the slope of a curve one could derive the differential formulas for the quadratic forms and thus study maxima and minima from a new point of view.

The next function in order is the integral rational function

$$y = a_0 + a_1x + a_2x^2 + \cdots a_nx^n.$$

It will first be considered in the simple power forms such as the formulas for volumes of cubes and spheres, or for areas under segments of the simple power curves. In connection with this

general function one might include such topics as polynomial multiplication and division, and the solution of integral rational equations by means of the factor theorem.

Some attention should now be given to the simplest fractional functions and irrational functions. In connection with these functions one would develop the notions of fractional and negative exponents. Now follow the exponential function, logarithms, and the geometric series.

During the first year the trigonometric ratios were introduced. These ratios should now be considered as functions. The various identities should be established and used in solving problems both of the plane and space.

I have not mentioned demonstrative geometry. I presume it should be offered in the high school but it does not fit in the general scheme I have outlined. I certainly should not give more time than one year to both the plane and solid. A large amount of the material presented in the usual course has little or no value since it is not used in the later mathematical work of the student.

Development of the Function Concept in the Mind of the Pupil. Evaluation of the formula. The first part of this paper has been concerned only with the arrangement of material. Let us now see what should be going on in the mathematical development of the pupil. At the beginning of the course he has had practically no experience in mathematical generalization. If he is given the sort of introduction I should give him he would start by translating statements expressed in ordinary language which go over into the simple linear type $y = ax$. The evaluation of y in such relations for various values of x leads him to see that y changes with x and in a very definite manner. Likewise the evaluation of x for various values of y leads to the conclusion that the converse statement holds. Since I have discussed quite fully the use of formulas in ninth grade work in the November number (1921) of the Mathematics Teacher and the development of the function concept in the June number (1921) of School Science and Mathematics I shall limit myself to only a few illustrations in this part of the paper.

Let us consider the formula,

$$h = d \tan \alpha$$

which is used in finding heights. For a fixed α the pupil finds that equal changes in d produce changes in h which are equal though different in value from the changes in d. He can easily discover for himself here that such a relation does not hold for all cases. For if the d is fixed and α varies he finds that equal increases in α do not give similar increases in h. Conversely if h remains fixed a similar statement holds for α and d. Thus, if h is the height of a pole and d is its shadow produced by the sun, his formulas tells him that doubling, or trebling, or quadrupling the angle of elevation does not give corresponding increases in the length of the shadow.

In connection with the consideration of the various types of functional relations the pupil should be given abundant practice in the translation of statements in ordinary language into formulas and in the evaluation of the letters in these formulas. It is this work in evaluation, which, if properly done, will lead him to discover for himself the nature of the various relations.

Thus let him evaluate F in the formula

$$F = 32 + 9/5\ C$$

for $C = -20, -10, 0, +10, +20$,
and evaluate C for $F = -13, -4, +5, +14$.
Let r mean the radius of a circular water-pipe. The area of its cross-section is given by the formula.

$$A = \pi r^2$$

Let him evaluate A for

$$r = 1,\ 2,\ 3,\ 4,\ \ldots$$

Then ask him how the area is affected by doubling the radius, trebling it, and quadrupling it. Had some city fathers, who recently had occasion to double the flow of water from a spring to their city, been given some mathematical training along this line they would not have doubled the diameter of the pipe-line.

As an example of how a pair of linear relations may work together consider the following problem:

Two trains, A and B, are moving east from a city, X. A is 25 miles east of X and is running at a uniform rate of 35 miles

per hour. B is 55 miles east of X and is running at a uniform rate of 30 miles per hour. Find the time when A will overtake B and their distance from X.

The problem may be solved by constructing a table of corresponding values of distances and time as follows:

t	0	1	2	3	4	5	6
d_1	25	60	95	130	165	200	235
d_2	55	85	115	145	175	205	235

As the table is constructed the children see that A is gaining on B and hence will over take B. If the two sets of distances are plotted against the corresponding values of the time and the graphs are constructed the relation between the distances is vividly portrayed.

Geometrical functions. In the course in demonstrative geometry as now generally taught the pupil meets functional relations which he cannot express analytically. For example he can see, although he is not able to state the precise relationship, that a side of a triangle, assuming that the other two sides are fixed, is a fraction of the angle opposite it. As the side grows the angle grows, and conversely. In a later course in trigonometry, or possibly in connection with the chapter on similarity, he may see that this relationship is expressed analytically by the law of cosines. At this time he can answer, and should be required to answer, such questions as the following. What is the length of the side if the angle is 0°? 90°? 180°? Does the side vary as the angle?

In his study of circles he finds that in a fixed circle the length of a chord depends on its distance from the center of the circle. As the distance grows the chord shrinks and conversely. But he cannot say whether the chord varies inversely as the distance, or according to some other law. After he has proved the Pythagorean theorem he can show that

$$C = 2\sqrt{r^2 - d^2}.$$

This formula enables him to answer questions he may have raised as to the nature of the relation.

The chapters on similarity and areas furnish numerous examples of geometrical relationships which can be expressed by simple formulas. These formulas, if studied as indicated above, give the pupil clearer notions as to the nature of these relationships. The character of these relationships may also in some cases be inferred from the geometrical figure. Thus if the sides of a rectangle are doubled it is easy to see that the enlarged rectangle has four times as many square units as the original rectangle.

The graph. A second instrument which is very useful in expressing relationship is the graph. By means of it the general character of the relationship can be seen at a glance. It can therefore be used for predicting the nature of the formula corresponding to it. As an illustration of this point let us plot values of weights used consecutively in stretching a rubber cord and the correcponding lengths of the cord. We find that the points lie approximately on a straight line. We construct a line which seems to fit the points best. After determining the y-intercept of the line and its slope we can write the formula which holds for this particular cord.

On account of its concreteness and its appeal to the eye the graph should be introduced early in the course. It should accompany or possibly precede the formula corresponding to it. The pupil should be able to associate with every type of formula with which he works the corresponding graph and conversely.

Some Reasons Why Functionality Should be Stressed. In the first paragraph of this paper I pointed out the fundamental character of the function concept. Later I showed how the function could be used in effecting the mechanical organization of the course. Finally I sketched a plan whereby the pupil might have his attention directed to the fact that mathematics and science are concerned with functional relations. I wish now to give some reasons why this should be done.

Variables. One of the words of common occurrence in mathematics, and which may be said to be the most distinctly mathematical of all notions,[1] is *variable.* A variable is sometimes

[1] Russell, Principles of Mathematics, p. 89.

defined as a symbol which in a given discussion may be used to denote any one of a given set of objects. Thus, in the statement, "A horse is a quadruped," the word, horse, is a variable since it is a symbol used to denote any one of a given set of objects. In particular it may be Black Beauty or Man of War.

In the formula,

$$A = \pi r^2,$$

A is a variable since it may refer to the area of any one of a class of objects called circles. r is a variable for a similar reason. Variable is therefore a word of very wide application. In fact the expression of a thought is an expression of relationship between variables.[1] The importance of the variable has been pointed out by Prof. Nunn[2] in the following statement. "The invention of variables was, perhaps, the most important event in human evolution. The command of their use remains the most significant achievement in the history of the individual human being. Ordinary algebra simply carries to a higher stage of usefulness in a special field the device which common language employs over the whole range of discourse. The prudent teacher will, therefore, in the interests of clear understanding and economy of effort, present the technical use of variables in mathematics not as a new thing but as merely a modification of linguistic uses which the pupil mastered, in principle, at his mother's knee."

Professor Judd, in his book on "The Psychology of High School Subjects" in the chapter on language writes: "The world of thought is enormously expanded by the creation and use of words. It is little wonder that man for long ages thought of himself as absolutely distinct from the animal kingdom. Man lives in a world of words; the animals live in a world of things and memories of things. To those who can use words so as to influence the rest of us we give society's great rewards. To the combination of ideas which have been worked out in words we owe changes which have been wrought out in things. In short, our civilization rests on words more than on things themselves, for our civilization differs from primitive uncouth con-

[1] For a fuller discussion of variables see G. A. Bliss on "The Function Concept and the Fundamental Notions of the Calculus" in "Monographs on Topics of Modern Mathematics" edited by J. W. A. Young.

[2] T. P. Nunn, "The Teaching of Algebra," p. 7.

ditions chiefly because the economical methods of thought and action made possible by words have transformed our relation to the world and put at man's disposal forces which could not have been discovered or mastered without the higher modes of abstract thought."

Now the higher modes of abstract thought are concerned with variables and relations between variables. Of course not all the higher modes of abstract thought are concerned with the variables which are employed in elementary mathematics. Here, for the most part, the variable is used to denote any one of a set of numbers. But even so its usefulness extends over a very broad field.

Now the variable of elementary mathematics is usually found in the formula which may always be considered as a functional relation between variables.

The formula. The formula is probably the most important instrument in the expression of mathematical thought. It may also be considered as a compact symbolic expression of a rule or principle. The great advances in mathematics date from the time of the introduction of symbols. Many mathematical formulations expressed in ordinary language would be bewildering and probably useless for practical purposes. Such a formula as Taylor's theorem, viz.,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \text{— ——},$$

is a good illustration. Above I quoted Prof. Judd as saying the "the economical methods of thought and action made possible by words have transformed our relation to the world and put at man's disposal forces which could not have been discovered or mastered without the higher modes of abstract thought." Now may we not say that if common language is an economizer of the first degree the language of mathematics as exemplified in the formula is an economizer of the second degree?

In this connection one writer[1] on philosophy makes the following statement. "The laws of physics, of astronomy, of geology, and the rest are, on the practical side, mere short hand, or rather shortmind, formulae or rules for recovering with a

[1] T. J. McCormack, "Why Do We Study Mathematics?"

minimum of mental labor by means of a little brains and the mechanical manipulation of a pencil, the past and the future facts of nature, which without these rules and laws, we should have indefinite labor, and take indefinite time in recovering. . . . Millions of cases reduce to a single case. All a saving of intellectual labor. But the economy of mental effort is most conspicuous in mathematics. . . . The student who has gained this point of view by contact with mathematical, and especially algebraic study, acquires from it a sense of intellectual power and discipline which can be brought home by no other branch of knowledge."

The importance of the formula can hardly be over emphasized. Formulas are found in great numbers in engineering and technical journals, in the sciences such as physics, chemistry, biology, geology, psychology, social science, pedagogy, in all sciences which have reached the statistical stage, in business, in a vast number of trades and professions in which a mathematical statement or investigation is required. In all of these fields they are used not only for expressing generalizations but for describing the relations between elements in these various fields, that is, for the expression of functionality.

Thus I find listed in a high school physics 64 formulas expressing relationships between various magnitudes in mechanics of solids, liquids, and gases, in sound, heat, electricity, and light. In a geometry I find a list of 101 formulas involving line values, angles, areas, and volumes. In a trigonometry I find listed 56 of the most important formulas pertaining to plane figures and 41 to spherical figures. In books and magazines on engineering I find formulas on almost every page. In a book on stastistical methods applied to education I find a list of 35 formulas. I could go on citing indefinitely from the fields of science illustrations of relationships expressed by means of formulas. In fact one science is considered as having reached a higher stage of development than another if the phenomena with which it is concerned can be explained by a mathematical formulation. Thus physics has reached a higher stage of development than chemistry. Physics made a great advance as a mathematical science when Newton formulated, after having examined the

mass of data collected laboriously by Copernicus, Tycho, Brahe, Kepler, Galileo, their predecessors and co-workers, that the attraction of two material bodies varies as the product of their masses and inversely as the square of the distance between them, i. e.

$$f = \frac{kmm'}{d^2}$$

In the same manner any science advances when a class of phenomena in its field can be explained by a mathematical formula. That is to say when this phenomena can be explained as a relationship between certain variables.

I have shown at some length that the notions of variable and functional relation are vitally related to the advancement of science. I may also add that the advancement of the ability of the individual to think depends on the skill he has attained in seeing and expressing relationships between variables.

Since scientific methods are being employed in more and more fields of human endeavor it is of prime importance that our secondary pupils be given the sort of mathematical training that will fit them for work in these fields. And the sort of mathematical training which will best fit them for the employment of scientific methods is that in which functionality is stressed.

In closing, I quote the following statement from the report of the National Committee on Mathematical Requirements,[1] a statement which should be kept constantly before teachers of secondary mathematics.

"The one great idea which is sufficient in scope to unify the course is that of the *functional relation.* The concept of a variable and of the dependence of one variable upon another is of fundamental importance for everyone. . . . The primary and underlying principle of the course, particularly in connection with algebra and trigonometry, should be the notion of relationship between variables including the methods of determining and expressing such relationship. The notion of relationship is fundamental both in algebra and geometry. The teacher should have it constantly in mind, and the pupils advancement should be consciously directed along the lines which will present first one and then another of the details upon which finally the formation of the general concept of functionality depends."

[1] U. S. Bureau of Education, Secondary School Circular No. 5, p. 3.